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Linear Ordinary Differential Equations with Gevrey Coefficients

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Ultradistribution solutions of Gevrey classes and distribution solutions are discussed for linear ordinary differential equations. The results previously established for equations with real analytic coefficients are extended to the case where the coefficients are in the corresponding Gevrey class or infinitely differentiable under a natural condition on the irregularity at each singular point of the equation.

We consider the linear differential operator

$$P(x, d/dx) = a_m(x)\, d^m/dx^m + a_{m-1}(x)\, d^{m-1}/dx^{m-1} + \cdots + a_0(x) \quad (0.1)$$

and the equation

$$P(x, d/dx)\, u(x) = f(x) \quad (0.2)$$

on an open interval Ω in $\mathbf{R}$.

When the coefficients $a_i(x)$ are real analytic functions on Ω with $a_m(x) \not\equiv 0$, the following results are known for hyperfunction solutions (Sato [17], Komatsu [3]).

THEOREM A. *For any $f \in \mathscr{B}(\Omega)$, (0.2) has a solution $u \in \mathscr{B}(\Omega)$.*

Here $\mathscr{B}(\Omega)$ denotes the space of all hyperfunctions on Ω.

THEOREM B. *The number of linearly independent solutions $u \in \mathscr{B}(\Omega)$ of the homogeneous equation $Pu = 0$ is equal to*

$$m + \sum_{x \in \Omega} \operatorname{ord}_x a_m(x), \quad (0.3)$$

where $\operatorname{ord}_x a_m(x)$ denotes the order of zeros of $a_m(x)$ at x.

THEOREM C. *If $f \in \mathscr{B}(\Omega)$, then any solution $u_1 \in \mathscr{B}(\Omega_1)$ of (0.2) on a subinterval Ω_1 of Ω can be continued to a solution $u \in \mathscr{B}(\Omega)$ on Ω.*

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We change the definition of [5] slightly and define the *irregularity* σ of a singular point x, i.e., a zero of $a_m(x)$, by

$$\sigma = \max \left\{ 1, \max_{0 \leqslant i < m} \frac{\operatorname{ord}_x a_m(x) - \operatorname{ord}_x a_i(x)}{m - i} \right\}. \tag{0.4}$$

Assuming again that the coefficients are real analytic, we have obtained in [5, 8] the following regularity theorems.

THEOREM D. *If $P(x, d/dx)$ has no singular points in Ω, then any solution $u \in \mathscr{B}(\Omega)$ of* (0.2) *belongs to $\mathscr{A}(\Omega)$ whenever $f \in \mathscr{A}(\Omega)$.*

Here $\mathscr{A}(\Omega)$ denotes the space of all real analytic functions on Ω.

THEOREM E. *If the irregularity σ is equal to* 1 *at every singular point in Ω, then any solution $u \in \mathscr{B}(\Omega)$ of* (0.2) *belongs to $\mathscr{D}'(\Omega)$ whenever $f \in \mathscr{D}'(\Omega)$.*

Here $\mathscr{D}'(\Omega)$ denotes the space of all distributions on Ω.

THEOREM F. *If the irregularity $\sigma \leqslant s/(s-1)$ at every singular point in Ω, then any solution $u \in \mathscr{B}(\Omega)$ of* (0.2) *belongs to $\mathscr{D}^{(s)\prime}(\Omega)$ whenever $f \in \mathscr{D}^{(s)\prime}(\Omega)$.*

Here $s > 1$ and $\mathscr{D}^{(s)\prime}(\Omega)$ is the space of all ultradistributions of class (s) on Ω.

THEOREM G. *If the irregularity $\sigma < s/(s-1)$, then any solution $u \in \mathscr{B}(\Omega)$ of* (0.2) *belongs to $\mathscr{D}^{\{s\}\prime}(\Omega)$ whenever $f \in \mathscr{D}^{\{s\}\prime}(\Omega)$.*

Here $s > 1$ and $\mathscr{D}^{\{s\}\prime}(\Omega)$ is the space of all ultradistributions of class $\{s\}$ on Ω.

Consequently if the irregularity condition of Theorem E (resp. Theorem F, resp. Theorem G) is satisfied, then Theorems A, B and C hold with $\mathscr{B}(\Omega)$ replaced by $\mathscr{D}'(\Omega)$ (resp. $\mathscr{D}^{(s)\prime}(\Omega)$, resp. $\mathscr{D}^{\{s\}\prime}(\Omega)$).

The purpose of this paper is to show that if the coefficients $a_i(x)$ belong to $\mathscr{E}(\Omega)$ and the irregularity σ is equal to 1 at every singular point (resp. $a_i(x) \in \mathscr{E}^{(s)}(\Omega)$ and $\sigma \leqslant s/(s-1)$, resp. $a_i(x) \in \mathscr{E}^{\{s\}}(\Omega)$ and $\sigma < s/(s-1)$), then Theorems A–G hold with $\mathscr{B}(\Omega)$ replaced by $\mathscr{D}'(\Omega)$ (resp. $\mathscr{D}^{(s)\prime}(\Omega)$, resp. $\mathscr{D}^{\{s\}\prime}(\Omega)$) and $\mathscr{A}(\Omega)$ by $\mathscr{E}(\Omega)$ (resp. $\mathscr{E}^{(s)}(\Omega)$, resp. $\mathscr{E}^{\{s\}}(\Omega)$).

Here $\mathscr{E}(\Omega)$ is the space of all infinitely differentiable functions on Ω, $\mathscr{E}^{(s)}(\Omega)$ (resp. $\mathscr{E}^{\{s\}}(\Omega)$) for $s > 1$ is the space of all ultradifferentiable functions of class (s) (resp. $\{s\}$) on Ω, i.e., all $\varphi \in \mathscr{E}(\Omega)$ such that for any compact set K in Ω and $h > 0$ there is a constant C (resp. there are constants h and C) for which

$$\sup_{x \in K} |\varphi^{(p)}(x)| \leqslant C h^p p!^s, \qquad p = 0, 1, 2, \ldots . \tag{0.5}$$

The space $\mathscr{D}^{(s)\prime}(\Omega)$ (resp. $\mathscr{D}^{\{s\}\prime}(\Omega)$) of ultradistributions of class (s) (resp. of class $\{s\}$) on Ω is by definition the space of all continuous linear functionals on the space $\mathscr{D}^{(s)}(\Omega)$ (resp. $\mathscr{D}^{\{s\}}(\Omega)$) of all $\varphi \in \mathscr{E}^{(s)}(\Omega)$ (resp. $\mathscr{E}^{\{s\}}(\Omega)$) with compact support endowed with a natural locally convex topology (see [4, 6, 12] for the theory of ultradistributions).

Our main theorem is Theorem 7 of Section 4. It gives the analogues of Theorems A, B and C. The counterpart of Theorem D is Theorem 1 of Section 1. Theorems E, F and G in our setting are formulated as Theorem 8 but it is only a direct consequence of the main theorem.

The proofs of Theorems A, B and C in [3] are based on Cauchy's existence theorem and the index formula of the analytic continuation $P(z, d/dz)$ of $P(x, d/dx)$ acting in the space $\mathscr{O}(V)$ of holomorphic functions on an open set V in $\mathbf{C}$ (see also Malgrange [13] and Komatsu [7] for the index formula).

Similarly we obtain our results from the existence theorem in the non-singular case (Lemma 2) and the index formulas of $P(x, d/dx)$ acting in various spaces of ultradifferentiable functions (Theorem 3). Section 3 is devoted to their proofs.

We also compute the index of $P(x, d/dx)$ acting in the space of formal power series with bounds (Theorem 4). This generalizes some index formulas of Malgrange [13] and Ramis [15] to the case where the coefficients of $P(x, d/dx)$ are not analytic. It implies together with a unique continuation theorem (Lemma 4) that the solvability of (0.2) in the space of ultradifferentiable functions near a singular point is equivalent to the formal solvability (Theorem 5).

We discuss only single differential equations for the sake of simplicity but it is not difficult to extend our results to systems of differential equations. For example, Methée's results [14] on the number of linearly independent homogeneous distribution solutions holds without analyticity of the coefficients.

1. Non-singular Case

We denote by $*$ one of $\varnothing$, (s) and $\{s\}$ for an $s > 1$. Thus $\mathscr{E}^*(\Omega)$ denotes the space of all infinitely differentiable functions, or all ultradifferentiable functions of class (s) or of class $\{s\}$ on Ω according as $* = \varnothing$ or (s) or $\{s\}$. $\mathscr{D}^*(\Omega)$ is its linear subspace of all functions with compact support.

In this section we assume that the coefficients $a_i(x)$ of $P(x, d/dx)$ are in $\mathscr{E}^*(\Omega)$ and that the leading coefficient $a_m(x)$ never vanishes on Ω.

Since the pointwise multiplication is a continuous bilinear mapping on $\mathscr{E}^*(\Omega) \times \mathscr{E}^*(\Omega)$ into $\mathscr{E}^*(\Omega)$ ([4, Theorem 2.8], cf. the proof of Lemma 2), the following lemma shows that we may assume $a_m(x) \equiv 1$.

LEMMA 1. *If $a(x) \in \mathscr{E}^*(\Omega)$ never vanishes, then $1/a(x)$ belongs to $\mathscr{E}^*(\Omega)$.*

Proof. This is well known when $* = \varnothing$. The result in the case $* = \{s\}$ is due to Rudin [16]. To prove it in general, we introduce the following notation. Let

$$A(X) = \sum_{p=0}^{\infty} \frac{A_p}{p!} X^p \tag{1.1}$$

be a formal power series in the indeterminate X with nonnegative coefficients A_p. We write

$$a(x) \ll_K A(X) \tag{1.2}$$

if for every p and $x \in K$

$$|a^{(p)}(x)| \leqslant A_p. \tag{1.3}$$

If $|a(x)| \geqslant l > 0$ on K, then we can easily prove that

$$1/a(x) \ll_K \sum_{q=0}^{\infty} l^{-q-1} \left(\sum_{p=1}^{\infty} A_p X^p/p! \right)^q. \tag{1.4}$$

Let $* = (s)$. For each compact set K in Ω and $h > 0$ we can find a constant C and an $l > 0$ such that

$$a(x) \ll_K C \sum_{p=0}^{\infty} h^p p!^{s-1} X^p$$

and $|a(x)| \geqslant l$ on K. Hence we have

$$\begin{aligned} 1/a(x) &\ll_K \sum_{q=0}^{\infty} l^{-q-1} \left(C \sum_{p=1}^{\infty} p!^{s-1} (hX)^p \right)^q \\ &= \sum_{r=0}^{\infty} B_r X^r/r!, \end{aligned}$$

where $B_0 = l^{-1}$ and

$$B_r = r!\, h^r \sum_{q=1}^{r} l^{-q-1} C^q \sum_{\substack{p_1+\cdots+p_q=r\\ p_i \geqslant 1}} p_1!^{s-1} \cdots p_q!^{s-1}, \qquad r \geqslant 1.$$

Here

$$\begin{aligned}\sum p_1!^{s-1} \cdots p_q!^{s-1} &\\ &\leqslant \sum ((r-q+1)!^{s-1})^{(p_1-1)/(r-q)} \cdots ((r-q+1)!^{s-1})^{(p_q-1)/(r-q)}\\ &= \binom{r-1}{q-1} (r-q+1)!^{s-1},\end{aligned}$$

and we can find a constant D such that

$$l^{-q-1} C^q \leqslant Dq!^{s-1}, \qquad q = 1, 2, \ldots .$$

Hence we have

$$\begin{aligned} B_r &\leqslant r!\, h^r \sum_{q=1}^{r} \binom{r-1}{q-1} Dr!^{s-1}\\ &= D 2^{r-1} h^r r!^s, \qquad r \geqslant 1,\end{aligned}$$

proving that $1/a(x) \in \mathscr{E}^{(s)}(\Omega)$.

The same proof works in the case $* = \{s\}$.

LEMMA 2. *Suppose that $a_m(x) = 1$ and $x_0 \in \Omega$. For each $f \in \mathscr{E}^*(\Omega)$ and $c_0, c_1, \ldots, c_{m-1} \in \mathbf{C}$ there is a unique solution $u \in \mathscr{E}^*(\Omega)$ of*

$$\begin{aligned} P(x, d/dx)\, u(x) &= f(x), & x &\in \Omega,\\ u^{(j)}(x_0) &= c_j, & j &= 0, 1, \ldots, m-1. \end{aligned} \tag{1.5}$$

Proof. Although we have more general results in [10, 11], we give a proof for the sake of completeness. The unique existence of a solution $u \in \mathscr{E}(\Omega)$ is well known. Therefore for each compact set K in Ω there are constants $B_0, B_1, \ldots, B_{m-1}$ depending on c_j and $f(x)$ such that

$$\sup_{x \in K} |u^{(j)}(x)| \leqslant B_j, \qquad j = 0, \ldots, m-1. \tag{1.6}$$

Suppose that $A(X) = \sum_{p=0}^{\infty} A_p X^p/p!$ and $F(X) = \sum_{p=0}^{\infty} F_p X^p/p!$ are formal power series such that

$$a_i(x) \ll_K A(X), \qquad f(x) \ll_K F(X). \tag{1.7}$$

If a formal power series $U(X) = \sum_{p=0}^{\infty} U_p X^p/p!$ satisfies

$$B_j \leqslant U_j, \qquad j = 0,\dots, m-1, \tag{1.8}$$

and

$$U^{(m)}(X) \gg \sum_{i=0}^{m-1} A(X)\, U^{(i)}(X) + F(X), \tag{1.9}$$

then we have

$$u(x) \ll_K U(X). \tag{1.10}$$

In fact, we obtain

$$\sup_{x \in K} |u^{(p)}(x)| \leqslant U_p, \qquad p = 0, 1, 2,\dots, \tag{1.11}$$

by induction on p. If $p \leqslant m-1$, this is immediate from (1.6) and (1.8). If $p = m$, then this follows from the equation

$$u^{(m)}(x) = -\sum_{i=0}^{m-1} a_i(x)\, u^{(i)}(x) + f(x), \tag{1.12}$$

(1.7) and (1.9). Suppose that (1.11) holds for $p < m + q$. Differentiating (1.12) q times, we have

$$\begin{aligned} \sup_{x \in K} |u^{(m+q)}(x)| &\leqslant \sup_{x \in K} \left| \frac{d^q}{dx^q} \left(-\sum_{i=0}^{m-1} a_i(x)\, u^{(i)}(x) + f(x) \right) \right| \\ &\leqslant \frac{d^q}{dX^q} \left(\sum_{i=0}^{m-1} A(X)\, U^{(i)}(X) + F(X) \right) \Bigg|_{X=0} \\ &\leqslant U^{(m+q)}(X)\,|_{X=0}. \end{aligned}$$

Thus (1.11) holds for $p = m + q$.

Now suppose that M_p, $p = 0, 1,\dots,$ is a sequence of positive numbers satisfying

$$M_p \leqslant M_{p+1}, \qquad p = 0, 1, 2,\dots, \tag{1.13}$$

$$\frac{M_p}{p!} \frac{M_q}{q!} \leqslant \frac{M_{p+q}}{(p+q)!}, \qquad p, q = 0, 1, 2,\dots. \tag{1.14}$$

We set

$$\theta(X) = \sum_{p=0}^{\infty} \frac{M_p}{p!} X^p. \tag{1.15}$$

Suppose that

$$a_i(x) \ll_K A(X) = B\theta(kX) \tag{1.16}$$

for some constants k and B. Then we claim that there is an $h_0 > k$ such that if $h \geqslant h_0$ then

$$U(X) = C\theta(hX) \tag{1.17}$$

satisfies (1.9) whenever

$$f(x) \ll_K F(X) = C\theta^{(m)}(hX). \tag{1.18}$$

First we note that

$$\theta^{(j)}(X) \ll \theta^{(j+1)}(X), \tag{1.19}$$

$$\theta(kX)\,\theta^{(j)}(hX) \ll (1-(k/h))^{-1}\,\theta^{(j)}(hX) \tag{1.20}$$

for any $j = 0, 1, 2,\ldots$ and $0 < k < h$. In fact, (1.19) is immediate from (1.13). Comparing the coefficients of $h^r X^r$ of both members of (1.20), we have

$$\sum_{p=0}^{r} \left(\frac{k}{h}\right)^p \frac{M_p}{p!}\frac{M_{r+j-p}}{(r-p)!} \leqslant \frac{1}{1-k/h}\frac{M_{r+j}}{r!}$$

because

$$\frac{M_p}{p!}\frac{M_{r+j-p}}{(r-p)!} \leqslant \frac{M_{r+j}}{(r+j)!}\frac{(r+j-p)!}{(r-p)!} \leqslant \frac{M_{r+j}}{r!}.$$

In view of these majorations we have by (1.16), (1.17) and (1.18)

$$\sum_{i=0}^{m-1} A(X)\,U^{(i)}(X) + F(X) \ll \left(CB(1-k/h)^{-1}\sum_{i=0}^{m-1} h^i + C\right)\theta^{(m)}(hX).$$

This is majorized by

$$U^{(m)}(X) = Ch^m\theta^{(m)}(hX)$$

if h is sufficiently large.

On the other hand, (1.8) holds if C is sufficiently large.

In case $* = \varnothing$, we take an arbitrary sequence M_p satisfying (1.13) and (1.14) such that (1.16) and (1.18) hold for some $0 < k < h$.

In case $* = \{s\}$, we may take

$$M_p = p!^s. \tag{1.21}$$

For each compact set K in Ω there are constants k and B such that (1.16) holds. Hence if $f(x)$ satisfies

$$\sup_{x\in K} |f^{(p)}(x)| \leqslant Ch^{p+m}(p+m)!^s$$

for sufficiently large h, then the solution $u(x)$ satisfies

$$\sup_{x\in K} |u^{(p)}(x)| \leqslant C_1 h^p p!^s$$

for a constant C_1.

In case $* = (s)$, we may take

$$M_p = h_1 h_2 \cdots h_p p!^s, \tag{1.22}$$

where h_p is a sequence of positive numbers converging to 0 as p tends to infinity.

In fact, let

$$N_p = \sup_{x\in K} \max\{a_0^{(p)}(x),..., a_{m-1}^{(p)}(x), f^{(p-m)}(x)\}.$$

Then we have

$$\sup_p \frac{N_p}{h^p p!^s} < \infty$$

for any $h > 0$. Hence there is a constant C and a sequence $1 \leqslant k_p \nearrow \infty$ such that

$$N_p \leqslant \frac{Cp!^s}{k_1 \cdots k_p}$$

(see [12, Lemma 3.4] for a proof). Define

$$h_p = \max\left\{k_p^{-1}, p^{-s}, \left(\frac{p-1}{p}\right)^{s-1} h_{p-1}\right\}. \tag{1.23}$$

Then it is easy to see that $h_p \to 0$. Clearly we have

$$N_p \leqslant CM_p,$$

so that (1.16) and (1.18) hold with $k=1$ and $h \geqslant 1$. Since

$$\frac{M_p}{M_{p-1}} = h_p p^s \geqslant 1,$$

$$\frac{M_{p-1}}{(p-1)!} \frac{p!^2}{M_p^2} \frac{M_{p+1}}{(p+1)!} = \frac{h_{p+1}(p+1)^{s-1}}{h_p p^{s-1}} \geqslant 1,$$

we have also (1.13) and (1.14).

This completes the proof of Lemma 2.

In particular, the solutions $u \in \mathscr{E}^*(\Omega)$ of the homogeneous equation

$$P(x, d/dx)\, u(x) = 0 \tag{1.24}$$

are parametrized by the initial values $u^{(j)}(x_0), j = 0,..., m-1$. Thus there are exactly m linearly independent solutions.

Moreover, the continuous surjection

$$P(x, d/dx)\colon \mathscr{E}^*(\Omega) \to \mathscr{E}^*(\Omega)$$

is a homomorphism by De Wilde's closed graph theorem [1]. This may also be proved directly. Actually the proof of Lemma 2 shows that the correspondence from f to the solution u with zero initial values is bounded on each bounded set. Hence it is continuous because $\mathscr{E}^*(\Omega)$ is a bornologic space.

Thus we have

PROPOSITION 1. *If $a_m(x)$ never vanishes on Ω, then*

$$0 \longrightarrow \mathbf{C}^m \longrightarrow \mathscr{E}^*(\Omega) \xrightarrow{P(x,d/dx)} \mathscr{E}^*(\Omega) \longrightarrow 0 \tag{1.25}$$

is a topologically exact sequence.

We define the formal dual $P'(x, d/dx)$ of $P(x, d/dx)$ as usual by

$$P'(x, d/dx)\, \varphi(x) = \sum_{i=0}^{m} \left(-\frac{d}{dx}\right)^i (a_i(x)\, \varphi(x)). \tag{1.26}$$

If $a_m(x) \neq 0$ on Ω, then the leading coefficient $(-1)^m a_m(x)$ of $P'(x, d/dx)$ does not vanish either.

PROPOSITION 2. *If $a_m(x)$ never vanishes on Ω, then*

$$0 \longrightarrow \mathscr{D}^*(\Omega) \xrightarrow{P'(x,d/dx)} \mathscr{D}^*(\Omega) \xrightarrow{Q} \mathbf{C}^m \longrightarrow 0 \tag{1.27}$$

is a topologically exact sequence. If $v_1,..., v_m$ are linearly independent solutions of (1.24) *in $\mathscr{E}^*(\Omega)$, then Q may be defined by $Q\varphi = (\langle \varphi, v_j \rangle)$.*

Proof. The uniqueness of Lemma 2 proves the injectivity of P'.

Let $K = [b, c]$ be a compact interval in Ω and denote by $\mathscr{D}_K^*$ the space of all $\varphi \in \mathscr{D}^*(\Omega)$ with $\operatorname{supp} \varphi \subset K$. Given a $\varphi \in \mathscr{D}_K^*$ let ψ be a solution of

$$P'(x, d/dx)\, \psi(x) = \varphi(x),$$
$$\psi^{(j)}(b) = 0, \qquad j = 0, 1,..., m-1.$$

ψ belongs to $\mathscr{D}_K^*$ if and only if $\psi^{(j)}(c) = 0$ for $j = 0, 1,..., m-1$. Those numbers depend on φ continuously by the proof of Lemma 2.

On the other hand, let $v_1, \ldots, v_m$ be linearly independent solutions of (1.24) in $\mathscr{E}^*(\Omega)$. Then they are linearly independent on int K and hence the linear functionals $\langle \varphi, v_j \rangle$ are linearly independent on $\mathscr{D}_K^*$ by the fundamental lemma of variational calculus. If $\varphi = P'(x, d/dx)\psi$ with a $\psi \in \mathscr{D}_K^*$, then

$$\begin{aligned} \langle \varphi, v_j \rangle &= \langle P'(x, d/dx)\psi, v_j \rangle \\ &= \langle \psi, P(x, d/dx)v_j \rangle = 0. \end{aligned}$$

Hence it follows that the conditions $\psi^{(j)}(c) = 0$, $j = 1, \ldots, m$, and $\langle \varphi, v_j \rangle = 0$, $j = 1, \ldots, m$ are equivalent, so that

$$0 \longrightarrow \mathscr{D}_K^* \xrightarrow{P'(x,d/dx)} \mathscr{D}_K^* \xrightarrow{Q} \mathbf{C}^m \longrightarrow 0$$

is topologically exact.

Taking the inductive limit of this sequence as K tends to Ω, we obtain the topological exactness of (1.27). This completes the proof of Proposition 2.

The dual

$$0 \longrightarrow \mathbf{C}^m \xrightarrow{Q'} \mathscr{D}^{*\prime}(\Omega) \xrightarrow{P(x,d/dx)} \mathscr{D}^{*\prime}(\Omega) \longrightarrow 0 \tag{1.28}$$

of (1.27) is also a topologically exact sequence of locally convex spaces. In particular, the kernel of $P(x, d/dx)$ in $\mathscr{D}^{*\prime}(\Omega)$, i.e., the space of all solutions u of (1.24) in $\mathscr{D}^{*\prime}(\Omega)$, coincides with the linear combinations of $v_1, \ldots, v_m$, i.e., the space of all solutions u of (1.24) in $\mathscr{E}^*(\Omega)$. Hence every solution u of (1.24) in $\mathscr{D}^{*\prime}(\Omega)$ belongs to $\mathscr{E}^*(\Omega)$.

If $f \in \mathscr{E}^*(\Omega)$, then there is a solution $u_0 \in \mathscr{E}^*(\Omega)$ of

$$P(x, d/dx)\, u(x) = f(x). \tag{1.29}$$

If $u \in \mathscr{D}^{*\prime}(\Omega)$ is an arbitrary solution of (1.29), then $u - u_0$ is in $\ker(P(x, d/dx)\colon \mathscr{D}^{*\prime}(\Omega) \to \mathscr{D}^{*\prime}(\Omega)) \subset \mathscr{E}^*(\Omega)$, so that u belongs to $\mathscr{E}^*(\Omega)$.

Thus we obtain the following.

THEOREM 1. *Suppose that the coefficients $a_i(x)$ of $P(x, d/dx)$ are in $\mathscr{E}^*(\Omega)$ and that the leading coefficient $a_m(x)$ never vanishes on Ω. Then for any $f \in \mathscr{D}^{*\prime}(\Omega)$ there is a solution $u \in \mathscr{D}^{*\prime}(\Omega)$ of* (1.29). *If $f \in \mathscr{E}^*(\Omega)$, then every solution $u \in \mathscr{D}^{*\prime}(\Omega)$ belongs to $\mathscr{E}^*(\Omega)$. In particular, we have*

$$\ker(P\colon \mathscr{D}^{*\prime}(\Omega) \to \mathscr{D}^{*\prime}(\Omega)) = \ker(P\colon \mathscr{E}^*(\Omega) \to \mathscr{E}^*(\Omega)). \tag{1.30}$$

This space of homogeneous solutions is m dimensional and a homogeneous solution $f(x)$ vanishes identically if the derivatives $f^{(j)}(x_0)$ vanish for $i = 0, 1, \ldots, m - 1$ at a point x_0 in Ω.

Lastly we consider formal solutions. We define the spaces $\mathscr{E}^*(0)$ of formal power series by

$$\mathscr{E}(0) = \left\{ \sum_{p=0}^{\infty} \varphi^{(p)} x^p/p!\,;\, \varphi^{(p)} \in \mathbf{C} \right\}, \tag{1.31}$$

$$\mathscr{E}^{(s)}(0) = \{\varphi \in \mathscr{E}(0);\, \forall h > 0\, |\varphi^{(p)}|/h^p p!^s \to 0\}, \tag{1.32}$$

$$\mathscr{E}^{\{s\}}(0) = \{\varphi \in \mathscr{E}(0);\, \exists h > 0\, |\varphi^{(p)}|/h^p p!^s \to 0\}. \tag{1.33}$$

These spaces have natural locally convex topologies as for $\mathscr{E}^*(\Omega)$ [6] Suppose that 0 is a point in Ω. Then

$$0 \longrightarrow \mathscr{E}^*(\Omega)^0 \longrightarrow \mathscr{E}^*(\Omega) \xrightarrow{\ \rho\ } \mathscr{E}^*(0) \longrightarrow 0 \tag{1.34}$$

is a topologically exact sequence, where ρ is the mapping of taking the formal Taylor series at 0 and $\mathscr{E}^*(\Omega)^0$ is the closed linear subspace of $\mathscr{E}^*(\Omega)$ consisting of all φ with $\varphi^{(p)}(0) = 0$ for all p [6, Theorem 4.4].

Now let $P(x, d/dx)$ be a linear differential operator of order m with coefficients $a_i(x)$ in $\mathscr{E}^*(\Omega)$. If $a_m(0) \neq 0$, then we can prove as above that

$$0 \longrightarrow \mathbf{C}^m \longrightarrow \mathscr{E}^*(0) \xrightarrow{P(x,d/dx)} \mathscr{E}^*(0) \longrightarrow 0 \tag{1.35}$$

is a topologically exact sequence. Another proof is obtained from the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{E}^*(\Omega)^0 & \longrightarrow & \mathscr{E}^*(\Omega) & \longrightarrow & \mathscr{E}^*(0) & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle P(x,d/dx)} & & \big\downarrow{\scriptstyle P(x,d/dx)} & & \big\downarrow{\scriptstyle P(x,d/dx)} & & \\ 0 & \longrightarrow & \mathscr{E}^*(\Omega)^0 & \longrightarrow & \mathscr{E}^*(\Omega) & \longrightarrow & \mathscr{E}^*(0) & \longrightarrow & 0 \end{array} \tag{1.36}$$

Here $P(x, d/dx)\colon \mathscr{E}^*(\Omega) \to \mathscr{E}^*(\Omega)$ is a differential operator whose coefficients $a_i(x) \in \mathscr{E}^*(\Omega)$ are extensions of the original coefficients $a_i(x) \in \mathscr{E}^*(0)$ by (1.34). We may assume that $a_m(x)$ never vanishes in Ω. Then

$$0 \longrightarrow \mathbf{C}^m \longrightarrow \mathscr{E}^*(\Omega) \xrightarrow{\ P\ } \mathscr{E}^*(\Omega) \longrightarrow 0$$

is exact by Proposition 2 and similarly the exactness of

$$0 \longrightarrow \mathscr{E}^*(\Omega)^0 \xrightarrow{\ P\ } \mathscr{E}^*(\Omega)^0 \longrightarrow 0 \tag{1.37}$$

is proved by Lemma 2. Hence the exactness of (1.35) follows from the *snake theorem* asserting that

$$\begin{aligned} 0 \longrightarrow \ker P \longrightarrow \ker Q \longrightarrow \ker R \\ \longrightarrow \operatorname{coker} P \longrightarrow \operatorname{coker} Q \longrightarrow \operatorname{coker} R \longrightarrow 0 \end{aligned} \tag{1.38}$$

is exact if

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] \arrow[d, "P"] & Y \arrow[r] \arrow[d, "Q"] & Z \arrow[r] \arrow[d, "R"] & 0 \\ 0 \arrow[r] & X_1 \arrow[r] & Y_1 \arrow[r] & Z_1 \arrow[r] & 0 \end{tikzcd} \tag{1.39}$$

is a commutative diagram with exact rows.

In our case the isomorphism

$$\ker(P: \mathscr{E}^*(\Omega) \to \mathscr{E}^*(\Omega)) \cong \ker(P: \mathscr{E}^*(0) \to \mathscr{E}^*(0))$$

is induced from the mapping $\rho: \mathscr{E}^*(\Omega) \to \mathscr{E}^*(0)$. Thus we obtain the following.

THEOREM 2. *Suppose that $P(x, d/dx)$ is a linear differential operator of order m with coefficients $a_i(x) \in \mathscr{E}^*(0)$ and $a_m(0) \neq 0$. Then for any $f \in \mathscr{E}^*(0)$ there is a solution $u \in \mathscr{E}^*(0)$ of*

$$P(x, d/dx)u = f. \tag{1.40}$$

There are exactly m linearly independent solutions $u \in \mathscr{E}^*(0)$ of

$$P(x, d/dx)u = 0. \tag{1.41}$$

If $P(x, d/dx): \mathscr{E}^*(0) \to \mathscr{E}^*(0)$ is the restriction of a linear differential operator $P(x, d/dx): \mathscr{E}^*(\Omega) \to \mathscr{E}^*(\Omega)$ of order m on an open interval $\Omega \ni 0$ such that $a_m(x) \neq 0$ on Ω and if $f \in \mathscr{E}^*(\Omega)$, then every solution u of (1.40) *in $\mathscr{E}^*(0)$ is uniquely extended to a solution of* (1.40) *in $\mathscr{E}^*(\Omega)$.*

2. THE IRREGULARITY CONDITION

From now on we consider operators with singular points. We assume, however, that the singular points, i.e., the zeros of $a_m(x)$, are isolated in Ω and of finite order.

We first restrict ourselves to the case where Ω has only one singular point 0. In the last Section 4 we obtain the results in the general case by pasting local results together.

The following inequality plays a very important role in this paper.

LEMMA 3. *Let $I = [a, b]$ be a compact interval containing* 0 *and let d and p be non-negative integers. If $\psi(x) \in C^{d+p}(I)$ has a zero at least of order d at* 0, *i.e.,*

$$\psi(0) = \psi'(0) = \cdots = \psi^{(d-1)}(0) = 0, \tag{2.1}$$

then the funtion $\varphi(x)$ *defined by*

$$\begin{aligned} \varphi(x) &= \psi(x)/x^d, \qquad && x \neq 0, \\ &= \psi^{(d)}(0)/d!, \qquad && x = 0, \end{aligned} \tag{2.2}$$

belongs to $C^p(I)$ *and satisfies*

$$\sup_{x\in I} |\varphi^{(p)}(x)| \leqslant \frac{p!}{(d+p)!} \sup_{x\in I} |\psi^{(d+p)}(x)| \tag{2.3}$$

and

$$\varphi^{(q)}(0) = \frac{q!}{(d+q)!}\, \psi^{(d+q)}(0), \qquad q = 0, 1,..., p. \tag{2.4}$$

Proof. We prove the lemma by induction on d. Let $d = 1$. Then we have by (2.1)

$$\begin{aligned} \varphi(x) &= \frac{\psi'(0)}{1!} + \frac{\psi''(0)}{2!} x + \cdots + \frac{\psi^{(p)}(0)}{p!} x^{p-1} \\ &\quad + \frac{1}{p!} \int_0^x \frac{(x-y)^p}{x} \psi^{(p+1)}(y)\, dy, \qquad x \neq 0. \end{aligned}$$

Since the integral term is of order x^p, $\varphi(x)$ belongs to $C^{p-1}(I)$ and satisfies (2.4) for $q < p$. If $x \neq 0$, then $\varphi(x)$ is clearly p times continuously differentiable and we have

$$\begin{aligned} \varphi^{(p)}(x) &= \frac{1}{p!} \int_0^x \frac{d^p}{dx^p} \left[\frac{(x-y)^p}{x}\right] \psi^{(p+1)}(y)\, dy \\ &= \int_0^x \frac{y^p}{x^{p+1}} \psi^{(p+1)}(y)\, dy. \end{aligned}$$

Since

$$|\varphi^{(p)}(x) - \psi^{(p+1)}(0)/(p+1)| \leqslant \int_0^x \left| \frac{y^p}{x^{p+1}} (\psi^{(p+1)}(y) - \psi^{(p+1)}(0)) \right| |dy|$$

converges to zero as x tends to 0, $\varphi(x)$ belongs to $C^p(I)$ and satisfies (2.4) for $q = p$.

Moreover, we have

$$\begin{aligned} |\varphi^{(p)}(x)| &\leqslant \int_0^x \left| \frac{y^p}{x^{p+1}} \psi^{(p+1)}(y) \right| |dy| \\ &\leqslant \frac{1}{p+1} \sup_{y\in I} |\psi^{(p+1)}(y)|. \end{aligned}$$

Suppose that the lemma holds for $d-1$ and that $\psi \in C^{d+p}(I)$ has a zero of order $\geqslant d$ at 0. Then

$$\begin{aligned} \chi(x) &= \psi(x)/x^{d-1}, && x \neq 0, \\ &= \psi^{(d-1)}(0)/(d-1)! = 0, && x = 0, \end{aligned} \tag{2.5}$$

belongs to $C^{p+1}(I)$ and satisfies

$$\sup_{x \in I} |\chi^{(p+1)}(x)| \leqslant \frac{(p+1)!}{(d+p)!} \sup_{x \in I} |\psi^{(d+p)}(x)|, \tag{2.6}$$

$$\chi^{(q)}(0) = \frac{q!}{(d+q-1)} \psi^{(d+q-1)}(0), \qquad q = 0, 1, \ldots, p+1. \tag{2.7}$$

$\chi(x)$ has a zero at 0 and $\varphi(x)$ defined by (2.2) satisfies

$$\begin{aligned} \varphi(x) &= \chi(x)/x, && x \neq 0, \\ &= \chi'(0), && x = 0. \end{aligned}$$

Consequently we have by the lemma for $d = 1$ and by (2.6) and (2.7)

$$\begin{aligned} \sup_{x \in I} |\varphi^{(p)}(x)| &\leqslant \frac{p!}{(p+1)!} \sup_{x \in I} |\chi^{(p+1)}(x)| \\ &\leqslant \frac{p!}{(d+p)!} \sup_{x \in I} |\psi^{(d+p)}(x)| \end{aligned}$$

and

$$\begin{aligned} \varphi^{(q)}(0) &= \frac{q!}{(q+1)!} \chi^{(q+1)}(0) \\ &= \frac{q!}{(d+q)!} \psi^{(d+q)}(0), \qquad q = 0, 1, \ldots, p+1. \end{aligned}$$

PROPOSITION 3. *If $a(x) \in \mathscr{E}^*(\Omega)$ has a zero at least of order d at 0, then*

$$\begin{aligned} b(x) &= a(x)/x^d, && x \neq 0, \\ &= a^{(d)}(0)/d!, && x = 0, \end{aligned}$$

belongs to $\mathscr{E}^*(\Omega)$.

Proof. $b(x)$ is infinitely differentiable by Lemma 3. Suppose that $a(x) \in \mathscr{E}^{(s)}(\Omega)$ (resp. $\mathscr{E}^{\{s\}}(\Omega)$). Then for any compact interval I in Ω and $h > 0$ there is a constant C (resp. there are constants h and C) such that

$$\sup_{x \in I} |a^{(p)}(x)| \leqslant Ch^p p!^s.$$

Hence we have by (2.3)'

$$\sup_{x \in I} |b^{(p)}(x)| \leqslant Ch^p p!(p+d)!^{s-1}.$$

Since $(p+d)!^{s-1}/p!^{s-1} \leqslant 2^{(s-1)p}A$ for a constant A, we have

$$\sup_{x \in I} |b^{(p)}(x)| \leqslant AC(2^{s-1}h)^p \, p!^s,$$

completing the proof of Proposition 3.

Suppose that $a_m(x)$ has a zero exactly of order d at 0 and that $a_i(x)$, $i = 0, 1, \ldots, m-1$, have a zero at least of order d_i at 0. Then we may assume that the operator has the form

$$P\left(x, \frac{d}{dx}\right) = x^d \frac{d^m}{dx^m} + b_{m-1}(x)\, x^{d_{m-1}} \frac{d^{m-1}}{dx^{m-1}} + \cdots + b_0(x)\, x^{d_0} \tag{2.8}$$

on a neighborhood Ω of 0 on which $a_m(x) \neq 0$ with coefficients $b_i(x) \in \mathscr{E}^*(\Omega)$.

We define the irregularity σ of a singular point x by (0.4). $\operatorname{ord}_x a_i(x)$ can be ∞ but (0.4) gives always a number $1 \leqslant \sigma \leqslant d$.

If σ is the irregularity of the singular point 0, then an invertible function times $P(x, d/dx)$ is of the form (2.8) with integers d_i such that

$$d_i \geqslant d - \sigma(m-i). \tag{2.9}$$

If $\sigma > 1$, then (2.9) is an equality for an index i.

We assume from now on that every singular point in Ω has the irregularity

$$\sigma = 1 \qquad \text{if} \quad * = \varnothing; \tag{2.10}$$

$$\sigma \leqslant \frac{s}{s-1} \qquad \text{if} \quad * = (s); \tag{2.11}$$

$$\sigma < \frac{s}{s-1} \qquad \text{if} \quad * = \{s\}. \tag{2.12}$$

We call this the *irregularity condition.*

LEMMA 4. *Suppose that $P(x, d/dx)$ has the unique singular point 0 in Ω and that the irregularity condition is satisfied. Then every solution $u \in \mathscr{E}^*(\Omega)^0$ of $P(x, d/dx)\, u(x) = 0$ vanishes identically.*

Proof. First we note that every $u(x) \in \mathscr{E}^*(\Omega)^0$ has the following estimate as x tends to 0:

$$u(x) = O(|x|^k) \qquad \text{for any } k \text{ if } * = \varnothing; \tag{2.13}$$

$$u(x) = O(\exp(-(1/h\,|x|)^{1/(s-1)})) \qquad \text{for any } h > 0 \text{ if } * = (s); \tag{2.14}$$

$$u(x) = O(\exp(-(1/h\,|x|)^{1/(s-1)})) \qquad \text{for some } h > 0 \text{ if } * = \{s\}. \tag{2.15}$$

In fact, (2.13) is clear from the Taylor expansion. To prove (2.14) and (2.15) suppose that

$$|u^{(p)}(x)| \leqslant Ch^p p!^s \tag{2.16}$$

on a compact interval I containing 0. Then, since

$$u(x) = \frac{1}{(p-1)!} \int_0^x (x-y)^{p-1}\, u^{(p)}(y)\, dy$$

for any p, we have

$$\begin{aligned} |u(x)| &\leqslant C \inf_p (h\,|x|)^p\, p!^{s-1} \\ &\leqslant C \exp\left(-\frac{s-1}{(h\,|x|)^{1/(s-1)}}\right), \qquad x \in I. \end{aligned}$$

Next to prove that $u(x) = 0$ for $x > 0$, we change the independent variable into

$$t = \log(1/x) \qquad \text{if} \quad \sigma = 1, \tag{2.17}$$

$$t = (1/x)^{\sigma - 1} \qquad \text{if} \quad \sigma > 1, \tag{2.18}$$

as in [5]. Let $w(t) = u(x(t))$. Then the column vector $\boldsymbol{w}(t) = {}^t(w(t), w'(t), \ldots, w^{(m-1)}(t))$ satisfies the equation

$$(d/dt - \boldsymbol{B}(t))\, \boldsymbol{w}(t) = 0,$$

where $\boldsymbol{B}(t)$ is an $m \times m$ matrix of uniformly bounded functions. Hence there is a constant M such that

$$\|\boldsymbol{w}(t)\| \leqslant e^{M|t - t_0|}\, \|\boldsymbol{w}(t_0)\| \tag{2.19}$$

in the Euclidean norm.

In case $* = \varnothing$ and $\sigma = 1$, we have by (2.13)

$$\| w(t_0) \| = O(e^{-kt_0})$$

for any k as $t_0 \to \infty$. Substitute this into (2.19) for a $k > M$ and let $t_0 \to \infty$. Then we have $w(t) = 0$ so that $u(x) = 0$ for $x > 0$.

The proofs in the other cases are similar because we have

$$(\sigma - 1)(s - 1) \leqslant 1 \qquad \text{if} \quad * = (s),$$

$$(\sigma - 1)(s - 1) < 1 \qquad \text{if} \quad * = \{s\}.$$

Remark. We did not use the ultradifferentiability of the coefficients $a_i(x)$ or $b_i(x)$ at all. Moreover, the assumption that $u \in \mathscr{E}^*(\Omega)^0$ is employed only in the form that $u^{(i)}(x)$, $i = 0, 1, \ldots, m-1$, satisfy (2.13) (or (2.15)) for a sufficiently large k (resp. for a sufficiently small h) determined by the equation. Hence the conclusion of the lemma holds if $a_i(x) \in \mathscr{E}(\Omega)$ and if $u \in C^k(\Omega)^0$ for a sufficiently large k wen $\sigma = 1$ or if $u(x)$ satisfies (2.16) for a sufficiently small h even when $\sigma = s/(s-1)$.

The results of the non-singular case have been derived from the topologically exact sequences (1.25), (1.27) and (1.35). We can not expect results so simple in the singular case. However, assume that 0 is the only singular point in the open interval Ω. Then the index of the operator $P(x, d/dx)$ or $P'(x, d/dx)$ is easily computed as it acts in each member of the following topologically exact sequences of locally convex spaces:

$$0 \longrightarrow \mathscr{E}^*(\Omega)^0 \longrightarrow \mathscr{E}^*(\Omega) \longrightarrow \mathscr{E}^*(0) \longrightarrow 0, \tag{2.20}$$

$$0 \longrightarrow \mathscr{D}^*_{\Omega_-}(\Omega) \longrightarrow \mathscr{D}^*(\Omega) \longrightarrow \mathscr{D}^*(\Omega_+) \longrightarrow 0. \tag{2.21}$$

Here $\mathscr{D}^*_{\Omega_-}(\Omega)$ is the closed linear subspace of $\mathscr{D}^*(\Omega)$ composed of all functions with support in $\Omega_- = \{x \in \Omega; x \leqslant 0\}$.

$\mathscr{D}^*(\Omega_+)$ is the space of all ultradifferentiable functions of class $*$ on $\Omega_+ = \{x \in \Omega; x \geqslant 0\}$ with compact support and endowed with a natural locally convex topology. The topological exactness of (2.21) follows from the Whitney type extension theorem of class $*$ [6, Theorem 4.4]. It says that $\mathscr{D}^*(\Omega_+)$ coincides with the space $\mathscr{D}^*(\Omega)|_{\Omega_+}$ of the restrictions of functions in $\mathscr{D}^*(\Omega)$ to Ω_+ endowed with the quotient topology.

The strong dual of (2.21) is the topologically exact sequence

$$0 \longrightarrow \mathscr{D}^{*\prime}_{\Omega_+}(\Omega) \longrightarrow \mathscr{D}^{*\prime}(\Omega) \longrightarrow \tilde{\mathscr{D}}^{*\prime}(\Omega_-) \longrightarrow 0. \tag{2.22}$$

Here $\mathscr{D}^{*\prime}_{\Omega_+}(\Omega)$ is the closed linear subspace of $\mathscr{D}^{*\prime}(\Omega)$ composed of all ultradistributions with support in Ω_+. $\tilde{\mathscr{D}}^{*\prime}(\Omega_-)$ is the space of all ultradistributions on $\Omega_- \backslash \{0\}$ which can be extended to ultradistributions on

Ω. It is endowed with the quotient topology. The elements will be called *extendable ultradistributions.*

Since $P'(x, d/dx)$ is injective in each space of the sequence (2.21), the index is equal to $-\dim \operatorname{coker} P'$. Hence it follows that $P(x, d/dx)$ acting in the dual is surjective and $\dim \ker P = \dim \operatorname{coker} P'$.

3. Index Formulas

In this section we assume that 0 is a unique singular point of a differential operator $P(x, d/dx)$ with the coefficients $a_i(x) \in \mathscr{E}^*(\Omega)$ defined on an open interval Ω. We also assume that the irregularity condition (2.10) or (2.11) or (2.12) is satisfied.

Our aim is to compute the index of $P(x, d/dx)$: $\mathscr{E}^*(\Omega) \to \mathscr{E}^*(\Omega)$ etc. As shown in Section 2 we may assume without loss of generality that

$$P\left(x, \frac{d}{dx}\right) = x^d \frac{d^m}{dx^m} + b_{m-1}(x)\, x^{d_{m-1}} \frac{d^{m-1}}{dx^{m-1}} + \cdots + b_1(x)\, x^{d_1} \frac{d}{dx} + b_0(x)\, x^{d_0}, \tag{3.1}$$

where $b_i(x) \in \mathscr{E}^*(\Omega)$ and d_i are integers satisfying

$$d - \sigma(m-i) \leqslant d_i < d, \qquad i = 0, 1, \ldots, m-1. \tag{3.2}$$

In order to apply the stability theorem of indices by Gohberg and Krein [2] we employ the representations

$$\mathscr{E}^*(\Omega) = \varprojlim_{K} \varinjlim_{M_p} E^{M_p}(K), \tag{3.3}$$

$$\mathscr{D}^*(\Omega) = \varinjlim_{K} \varinjlim_{M_p} D_K^{M_p}, \tag{3.4}$$

$$\mathscr{D}^*(\Omega_+) = \varinjlim_{K} \varinjlim_{M_p} D_K^{M_p}(\Omega_+), \tag{3.5}$$

where K ranges over the compact intervals in Ω and $E^{M_p}(K)$, $D_K^{M_p}$ and $D_K^{M_p}(\Omega_+)$ are Banach spaces defined by

$$E^{M_p}(K) = \{\varphi \in C^\infty(K);\ \sup |\varphi^{(p)}(x)|/M_p \to 0 \text{ as } p \to \infty\}, \tag{3.6}$$

$$D_K^{M_p} = \{\varphi \in C^\infty(\mathbf{R});\ \operatorname{supp} \varphi \subset K,\ \sup |\varphi^{(p)}(x)|/M_p \to 0 \text{ as } p \to \infty\}, \tag{3.7}$$

$$D_K^{M_p}(\Omega_+) = \{\varphi \in C^\infty(\Omega_+);\ \operatorname{supp} \varphi \subset K \cap \Omega_+,\ \sup |\varphi^{(p)}(x)|/M_p \to 0 \text{ as } p \to \infty\}. \tag{3.8}$$

We choose a directed set of sequences M_p of positive numbers so that the balls of the Banach spaces form a fundamental system of bounded sets of $\mathscr{E}^*(K)$ or $\mathscr{D}_K^*$ or $\mathscr{D}_K^*(\Omega_+)$. Then, since those spaces are ultrabornologic [4, Theorem 2.6], (3.3), (3.4) and (3.5) hold topologically.

In case $* = \varnothing$, we take all sequences M_p satisfying (1.13), (1.14) and

$$\sup_{\substack{x \in K \\ i}} |b_i^{(p)}(x)| \leqslant B 2^{-p} M_p \tag{3.9}$$

for a constant B and such that there is a sequence N_p satisfying the Denjoy–Carleman condition and

$$k^p N_{p+i}/M_p \to 0 \qquad \text{as} \quad p \to \infty \tag{3.10}$$

for any $k > 0$ and $i = 0, 1, 2, \ldots$.

Given a sequence $L_p > 0$ we can easily find a sequence M_p satisfying (1.13), (1.14), (3.9) and

$$\max\{L_p, p!^s\} \leqslant C M_p, \qquad p = 0, 1, 2, \ldots,$$

for an $s > 1$ and C. Then (3.10) is satisfied for $N_p = p!^{s'}$ if $1 < s' < s$. Clearly those M_p form a directed set. Hence they satisfy our requirements.

In case $* = \{s\}$, we take all

$$M_p = h^p p!^s \tag{3.11}$$

such that (3.9) holds. This is the case if h is sufficiently large. Clearly M_p satisfies (1.13) and (1.14). We have also (3.10) for $N_p = p!^{s'}$ if $1 < s' < s$.

Lastly in case $* = (s)$, we take all sequences M_p satisfying (1.13), (1.14), (3.9) and (3.10) for $N_p = p!^{s'}$ with a fixed $0 < s' < s$ and of the form

$$M_p = h_1 h_2 \cdots h_p p!^s \tag{3.12}$$

with a sequence $h_p > 0$ tending to 0 as $p \to \infty$. If we change (1.23) in the proof of Lemma 2 slightly and define h_p by

$$h_p = \max \left\{ k_p^{-1}, p^{t-s}, \left(\frac{p-1}{p}\right)^{s-1} h_{p-1} \right\}$$

with a t such that $s' < t < s$, then we find that those sequences M_p form a fundamental system of sequences $L_p > 0$ such that for any $h > 0$

$$L_p/(h^p p!^s) \to 0 \qquad \text{as} \quad p \to 0.$$

Hence we have the representations (3.3), (3.4) and (3.5).

Thus we assume from now on that M_p is a sequence satisfying (1.13), (1.14), (3.9) and (3.10) for a sequence N_p with the Denjoy–Carleman condition.

Define the formal power series $\theta(X)$ by (1.15). Then (3.9) is equivalent to

$$b_i(x) \ll_K B\theta(2^{-1}X), \qquad i = 0, 1,..., m-1.$$

Hence it follows from (1.20) that the multiplication by $b_i(x)$ is a continuous linear operator in $E^{M_p}(K)$, in $D_K^{M_p}$ and in $D_K^{M_p}(\Omega_+)$.

We define as in [4] the spaces $\mathscr{E}^{\{N_p\}}(K)$, $\mathscr{D}_K^{\{N_p\}}$ and $\mathscr{D}_K^{\{N_p\}}(\Omega_+)$ by

$$\mathscr{E}^{\{N_p\}}(K) = \varinjlim_{h \to \infty} E^{h^p N_p}(K), \tag{3.13}$$

$$\mathscr{D}_K^{\{N_p\}} = \varinjlim_{h \to \infty} D_K^{h^p N_p}, \tag{3.14}$$

$$\mathscr{D}_K^{\{N_p\}}(\Omega_+) = \varinjlim_{h \to \infty} D_K^{h^p N_p}(\Omega_+). \tag{3.15}$$

LEMMA 5. *$\mathscr{E}^{\{N_p\}}(K)$ (resp. $\mathscr{D}_K^{\{N_p\}}$, resp. $\mathscr{D}_K^{\{N_p\}}(\Omega_+)$) is a dense linear subspace of $E^{M_p}(K)$ (resp. $D_K^{M_p}$, resp. $D_K^{M_p}(\Omega_+)$) such that the derivatives of its elements are in the latter space.*

Proof. We may assume without loss of generality that K is a compact interval $[a, b]$ with $a < 0 < b$. If $\varphi(x) \in E^{M_p}(K)$, then its dilation $\varphi((1-\varepsilon)x)$ belongs to $E^{M_p}((1-\varepsilon)^{-1} K)$ and tends to $\varphi(x)$ in $E^{M_p}(K)$ as $\varepsilon \to 0$. Let $j(x)$ be a function in $\mathscr{D}^{\{N_p\}}(\mathbf{R})$ with support in the unit interval and such that $\int j(x)\,dx = 1$. Then the regularization

$$\delta^{-1} \int j(\delta^{-1}y)\, \varphi((1-\varepsilon)(x-y))\, dy$$

belongs to $\mathscr{E}^{\{N_p\}}(K)$ for sufficiently small $\delta > 0$ and tends to $\varphi((1-\varepsilon)x)$ in $E^{M_p}(K)$ as $\delta \to 0$.

If ψ is a function in $\mathscr{E}^{\{N_p\}}(K)$, then the derivatives $\psi^{(i)}$ belong to $E^{M_p}(K)$ by (3.10).

If we consider the translation $\varphi(x+\varepsilon)$ instead of the dilation, the proof goes as well for functions φ in $D_K^{M_p}(\Omega_+)$.

In the case of $D_K^{M_p}$ we take ultradifferentiable functions χ_1 and χ_2 of class $\{N_p\}$ on $\mathbf{R}$ such that $\chi_1(x) + \chi_2(x) \equiv 1$, $\operatorname{supp} \chi_1 \subset (-\infty, d]$ and $\operatorname{supp} \chi_2 \subset [c, \infty)$ with $a < c < d < b$. If φ is a function in $D_K^{M_p}$, then $\chi_i \varphi$ belong to $D_K^{M_p}$ by (1.20) and (3.10). The translations $\chi_1(x-\varepsilon)\,\varphi(x-\varepsilon)$ and $\chi_2(x+\varepsilon)\,\varphi(x+\varepsilon)$ have compact support in the interior of K for sufficiently $\varepsilon > 0$ and converge to $\chi_1\varphi$ and $\chi_2\varphi$ in $D_K^{M_p}$. Their regularizations belong to $\mathscr{D}_K^{\{N_p\}}$ and converge to $\chi_1\varphi$ and $\chi_2\varphi$ in $D_K^{M_p}$, respectively.

PROPOSITION 4. *If K is a compact interval, then*

$$d/dx: E^{M_p}(K) \to E^{M_p}(K),$$
$$d/dx: D_K^{M_p} \to D_K^{M_p},$$
$$d/dx: D_K^{M_p}(\Omega_+) \to D_K^{M_p}(\Omega_+)$$

are closed linear operators with dense domains. Moreover, these operators have the following indices:

$$\operatorname{index}(d/dx: E^{M_p}(K) \to E^{M_p}(K)) = 1, \tag{3.16}$$

$$\operatorname{index}(d/dx: D_K^{M_p} \to D_K^{M_p}) = -1, \tag{3.17}$$

$$\operatorname{index}(d/dx: D_K^{M_p}(\Omega_+) \to D_K^{M_p}(\Omega_+)) = 0. \tag{3.18}$$

Proof. The domains are dense by Lemma 5. Clearly we have

$$\ker(d/dx: E^{M_p}(K) \to E^{M_p}(K)) = \mathbf{C}, \tag{3.19}$$

$$\ker(d/dx: D_K^{M_p} \to D_K^{M_p}) = 0, \tag{3.20}$$

$$\ker(d/dx: D_K^{M_p}(\Omega_+) \to D_K^{M_p}(\Omega_+)) = 0. \tag{3.21}$$

If $\psi \in E^{M_p}(K)$, then

$$I\psi(x) = \int_0^x \psi(y)\, dy$$

gives a continuous right inverse of d/dx by condition (1.13). Hence d/dx in $E^{M_p}(K)$ is a closed linear operator with index 1.

If $\psi = d\varphi/dx$ is in the image of $d/dx: D_K^{M_p} \to D_K^{M_p}$, then

$$\int_a^b \psi(x)\, dx = \varphi(b) - \varphi(a) = 0.$$

Conversely if $\psi \in D_K^{M_p}$ satisfies $\int \psi(x)\, dx = 0$, then

$$\varphi(x) = \int_a^x \psi(y)\, dy$$

belongs to $D_K^{M_p}$ and we have $\psi = d\varphi/dx$. Since the mapping $\psi \mapsto \varphi$ is continuous, $d/dx: D_K^{M_p} \to D_K^{M_p}$ is a closed linear operator with index -1.

The proof is similar for $d/dx: D_K^{M_p}(\Omega_+) \to D_K^{M_p}(\Omega_+)$.

PROPOSITION 5. *Suppose that K is a compact interval containing* 0. *Then*

$$x\frac{d}{dx}: E^{M_p}(K) \to E^{M_p}(K),$$

$$x\frac{d}{dx}: D_K^{M_p} \to D_K^{M_p},$$

$$x\frac{d}{dx}: D_K^{M_p}(\Omega_+) \to D_K^{M_p}(\Omega_+)$$

are closed linear operators with dense domains and with the following indices:

$$\text{index}\left(x\frac{d}{dx}: E^{M_p}(K) \to E^{M_p}(K)\right) = 0, \tag{3.22}$$

$$\text{index}\left(x\frac{d}{dx}: D_K^{M_p} \to D_K^{M_p}\right) = -2, \tag{3.23}$$

$$\text{index}\left(x\frac{d}{dx}: D_K^{M_p}(\Omega_+) \to D_K^{M_p}(\Omega_+)\right) = -1. \tag{3.24}$$

Proof. Clearly we have

$$\operatorname{dom}(x\,d/dx) \supset \operatorname{dom}(d/dx), \tag{3.25}$$

$$\ker(x\,d/dx) = \ker(d/dx) \tag{3.26}$$

in each space.

If ψ is in the image of $x\,d/dx: E^{M_p}(K) \to E^{M_p}(K)$, it must satisfy $\psi(0) = 0$. Conversely if $\psi \in E^{M_p}(K)$ satisfies $\psi(0) = 0$, then the function $\psi(x)/x$ is infinitely differentiable by Lemma 3 and its primitive $\varphi(x) = \int_0^x (\psi(y)/y)\,dy$ satisfies

$$\sup_{x \in K} |\varphi^{(p+1)}(x)| \leqslant \frac{1}{p+1} \sup_{x \in K} |\psi^{(p+1)}(x)| \tag{3.27}$$

for $p = 0, 1, 2, \ldots$. Since

$$\sup_{x \in K} |\varphi(x)| \leqslant |K| \sup_{x \in K} |\varphi'(x)|,$$

the mapping $\psi \mapsto \varphi$ is continuous. Hence $x\,d/dx: E^{M_p}(K) \to E^{M_p}(K)$ is a closed linear operator with image of codimension 1.

Similarly a necessary and sufficient condition in order that $\psi \in D_K^{M_p}$ (resp. $D_K^{M_p}(\Omega_+)$) be in the image of $x\,d/dx\colon D_K^{M_p} \to D_K^{M_p}$ (resp. $x\,d/dx\colon D_K^{M_p}(\Omega_+) \to D_K^{M_p}(\Omega_+)$) is

$$\psi(0) = \int ((\psi(x) - \psi(0))/x)\,dx = 0 \quad \text{(resp. } \psi(0) = 0\text{)}.$$

The expression

$$\varphi(x) = -\int_x^b (\psi(y) - \psi(0))/y\,dy + \chi(x) \int (\psi(y) - \psi(0))/y\,dy$$

is a continuous left inverse if χ is an ultradifferentiable function of class $\{N_p\}$, which is equal to 1 for $x \leqslant a + \varepsilon$ and equal to 0 for $x \geqslant -\varepsilon$ with an $\varepsilon > 0$.

We recall that a linear operator B from a Banach space X into a Banach space Y is said to be A-compact for a closed linear operator $A\colon X \to Y$ if B is compact as a linear operator from $\operatorname{dom} A$ with the norm $\|x\| + \|Ax\|$ into Y.

Proposition 6. *The identity mapping 1 is (d/dx)-compact and $(x\,d/dx)$-compact in $E^{M_p}(K)$, in $D_K^{M_p}$ and in $D_K^{M_p}(\Omega_+)$.*

Proof. Since $\|x(d/dx)\varphi\| \leqslant C\,\|(d/dx)\varphi\|$ with a constant C, we have only to prove that the set B of all φ in $E^{M_p}(K)$, etc., such that

$$\sup_{x \in K} |\varphi^{(p)}(x)| \leqslant M_p, \qquad p = 0, 1, \ldots, \tag{3.28}$$

$$\sup_{x \in K} |(d/dx)^p (x\varphi'(x))| \leqslant M_p, \qquad p = 0, 1, \ldots, \tag{3.29}$$

is precompact.

In view of Lemma 3, (3.29) implies

$$\sup_{x \in K} |\varphi^{(p+1)}(x)| \leqslant M_{p+1}/(p+1), \qquad p = 0, 1, \ldots. \tag{3.30}$$

Given an $\varepsilon > 0$, we take a $p_0 > 2\varepsilon^{-1}$. By the Ascoli–Arzela theorem there are a finite number of $\varphi_j \in B$ such that for any $\varphi \in B$

$$\sup_{x \in K} |\varphi^{(p)}(x) - \varphi_j^{(p)}(x)| \leqslant \varepsilon M_p, \qquad p = 0, 1, \ldots, p_0,$$

for some j. Then we have

$$\|\varphi - \varphi_j\| = \sup_{\substack{x \in K \\ p}} |\varphi^{(p)}(x) - \varphi_j^{(p)}(x)|/M_p \leqslant \varepsilon.$$

PROPOSITION 7. *Let K be a compact interval containing* 0 *and let* $0 \leqslant d \leqslant m$ *be integers. Then*

$$x^d \frac{d^m}{dx^m} : E^{M_p}(K) \to E^{M_p}(K),$$

$$x^d \frac{d^m}{dx^m} : D_K^{M_p} \to D_K^{M_p},$$

$$x^d \frac{d^m}{dx^m} : D_K^{M_p}(\Omega_+) \to D_K^{M_p}(\Omega_+)$$

are closed linear operator with dense domains and with the indices

$$\operatorname{index}\left(x^d \frac{d^m}{dx^m} : E^{M_p}(K) \to E^{M_p}(K)\right) = m - d, \tag{3.31}$$

$$\operatorname{index}\left(x^d \frac{d^m}{dx^m} : D_K^{M_p} \to D_K^{M_p}\right) = -m - d, \tag{3.32}$$

$$\operatorname{index}\left(x^d \frac{d^m}{dx^m} : D_K^{M_p}(\Omega_+) \to D_K^{M_p}(\Omega_+)\right) = -d. \tag{3.33}$$

Proof. It is easy to prove by induction on d that

$$x^d \frac{d^m}{dx^m} = \left(x \frac{d}{dx} - (d-1)\right)\left(x \frac{d}{dx} - (d-2)\right) \cdots \left(x \frac{d}{dx} - 1\right) \times \left(x \frac{d}{dx}\right) \frac{d^{m-d}}{dx^{m-d}}.$$

Here $x\, d/dx - k$ is a closed linear operator with the same domain and index as $x\, d/dx$ by the stability theorem of index [2, Theorem 2.6]. Hence it follows from the additive formula of index [2, Theorem 2.1] that the product $x^d (d/dx)^m$ is a closed linear operator with dense domain and with the index

$$\operatorname{index}\left(x^d \frac{d^m}{dx^m}\right) = d \operatorname{index}\left(x \frac{d}{dx}\right) + (m-d) \operatorname{index}\left(\frac{d}{dx}\right).$$

Consequently the index is computed by Propositions 5 and 6. A direct proof is not difficult either.

PROPOSITION 8. *Let K, m and d be as in Proposition* 7.

(i) *If* $0 \leqslant i < m$ *and*

$$d - (m - i) \leqslant c < d, \tag{3.34}$$

then $x^c\, d^i/dx^i$ *is* $(x^d\, d^m/dx^m)$*-compact in* $E^{M_p}(K)$, $D_K^{M_p}$ *and* $D_K^{M_p}(\Omega_+)$.

(ii) *Suppose that* M_p *is given by* (3.11) (*resp. by* (3.12) *with* $h_p \to 0$). *If* $0 \leqslant i < m$ *and*

$$d - \frac{s}{s-1}(m-i) < c < d \tag{3.35}$$

or, respectively,

$$d - \frac{s}{s-1}(m-i) \leqslant c < d, \tag{3.36}$$

then $x^c\, d^i/dx^i$ *is* $(x^d\, d^m/dx^m)$*-compact in* $E^{M_p}(K)$, $D_K^{M_p}$ *and* $D_K^{M_p}(\Omega_+)$.

Proof. Let B be the set of all $\varphi \in E^{M_p}(K)$ etc. such that

$$\sup_{x\in K} |\varphi^{(p)}(x)| \leqslant M_p, \qquad p = 0, 1, \ldots, \tag{3.37}$$

$$\sup_{x\in K} |(d/dx)^p (x^d \varphi^{(m)}(x))| \leqslant M_p, \qquad p = 0, 1, \ldots . \tag{3.38}$$

We have to prove that $(x^c\, d^i/dx^i)B$ is precompact.

Denote $x^d \varphi^{(m)}(x)$ by $\psi(x)$. Then we have

$$\begin{aligned} &\left(\frac{d}{dx}\right)^{m-i+c} (x^c \varphi^{(i)}(x)) \\ &= \sum_{q=0}^{c} \binom{m-i+c}{c-q} \left[\left(\frac{d}{dx}\right)^{c-q} x^c\right]\left[\left(\frac{d}{dx}\right)^{q}\left(\frac{\psi(x)}{x^d}\right)\right] \\ &= \sum_{q=0}^{c} A_q \left(\frac{d}{dx}\right)^{q}\left(\frac{\psi(x)}{x^{d-q}}\right) \end{aligned}$$

with constants A_q because $x^q (d/dx)^q$ is equal to a linear combination of $(d/dx)^q\, x^q$, $(d/dx)^{q-1}\, x^{q-1}, \ldots, 1$. Hence it follows from (2.3) and (3.38) that

$$\begin{aligned} &\sup_{x\in K} |(d/dx)^p (x^c \varphi^{(i)}(x))| \\ &\leqslant \sum_{q=0}^{c} |A_q| \frac{(p-m+i-c+q)!}{(p-m+i-c+d)!} M_{p-m+i-c+d} \\ &\leqslant A \frac{(p-m+i)!}{(p-m+i-c+d)!} M_{p-m+i-c+d} \end{aligned} \tag{3.39}$$

for $p \geqslant m - i + c$ with a constant A independent of p.

If (3.34) holds, then the right hand side of (3.39) is bounded by $A\, M_p/(p-m+i+1)$. Therefore the precompactness of $(x^c\, d^i/dx^i)B$ is proved in the same way as Proposition 6.

Next suppose that $M_p = h^p p!^s$ and (3.35) holds. If $m - i + c - d \geqslant 0$, then (i) applies. Hence we may assume that

$$0 > m - i + c - d > -(d - c)/s.$$

Then (3.39) is bounded by

$$\begin{aligned} &A \frac{h^{-m+i-c+d}(p - m + i - c + d)^{(-m+i-c+d)s}}{(p - m + i + 1)^{d-c}} M_p \\ &\leqslant \frac{A' h^{-m+i-c+d}}{(p - m + i - c + d)^{d-c+(m-i+c-d)s}} M_p \\ &= o(M_p). \end{aligned}$$

Lastly suppose that $M_p = h_1 \cdots h_p p!^s$ with $h_p \to 0$ and (3.36) holds. We may assume that

$$0 > m - i + c - d \geqslant -(d - c)/s.$$

Hence (3.39) is bounded by

$$A' h_{p+1} \cdots h_{p-m+i-c+d} M_p = o(M_p).$$

Now we are able to prove the index formulas of differential operators.

THEOREM 3. *Suppose that $P(x, d/dx)$ is a differential operator of order m with coefficients $a_i(x) \in \mathscr{E}^*(\Omega)$ on an open interval Ω and that $0 \in \Omega$ is a unique singular point satisfying the irregularity condition* (2.10) *or* (2.11) *or* (2.12). *Then $P(x, d/dx)\colon \mathscr{E}^*(\Omega) \to \mathscr{E}^*(\Omega)$ is a topological homomorphism with the index*

$$\operatorname{index}(P\colon \mathscr{E}^*(\Omega) \to \mathscr{E}^*(\Omega)) = m - \operatorname{ord}_0 a_m(x) \tag{3.40}$$

and

$$0 \longrightarrow \mathscr{D}^*(\Omega) \xrightarrow{P(x,d/dx)} \mathscr{D}^*(\Omega) \longrightarrow \mathbf{C}^{m+\operatorname{ord}_0 a_m(x)} \longrightarrow 0, \tag{3.41}$$

$$0 \longrightarrow \mathscr{D}^*(\Omega_+) \xrightarrow{P(x,d/dx)} \mathscr{D}^*(\Omega_+) \longrightarrow \mathbf{C}^{\operatorname{ord}_0 a_m(x)} \longrightarrow 0 \tag{3.42}$$

are topologically exact.

Proof. We may assume without loss of generality that $P(x, d/dx)$ has the form (3.1) with $b_i(x) \in \mathscr{E}^*(\Omega)$ and integers d_i satisfying (3.2).

First we assume that $d(=\operatorname{ord}_0 a_m(x))$ is less than or equal to m.

Then the leading term $x^d(d/dx)^m$ is an operator with indices (3.31), (3.32) and (3.33) in $E^{M_p}(K)$, $D_K^{M_p}$ and $D_K^{M_p}(\Omega_+)$, respectively. The irregularity condition together with (3.2) implies that $c = d_i$ satisfies the condition of Proposition 8. The multiplication by $b_i(x)$ is continuous in each space by (3.9) and (1.20). Hence $P(x, d/dx) - x^d(d/dx)^m$ is $(x^d(d/dx)^m)$-compact. Consequently $P(x, d/dx)\colon E^{M_p}(K) \to E^{M_p}(K)$, etc., is a closed linear operator with the same index as $x^d(d/dx)^m$.

Now let $L_p \geqslant M_p$ be another sequence satisfying (1.13), (1.14), (3.9) and (3.10) and of the form (3.11) if $* = \{s\}$ or (3.12) if $* = (s)$. In the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \ker P^{M_p} \arrow[r] \arrow[d] & \operatorname{dom} P^{M_p} \arrow[r, "P(x,d/dx)"] \arrow[d, hook] & E^{M_p}(K) \arrow[r] \arrow[d, hook] & \operatorname{coker} P^{M_p} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \ker P^{L_p} \arrow[r] & \operatorname{dom} P^{L_p} \arrow[r, "P(x,d/dx)"] & E^{L_p}(K) \arrow[r] & \operatorname{coker} P^{L_p} \arrow[r] & 0 \end{tikzcd}$$

the imbedding $E^{M_p}(K) \to E^{L_p}(K)$ has a dense image because it contains the dense subspace $\mathscr{E}^{\{N_p\}}(K)$. Hence the mapping $\operatorname{coker} P^{M_p} \to \operatorname{coker} P^{L_p}$ is surjective. On the other hand, the mapping $\ker P^{M_p} \to \ker P^{L_p}$ is clearly injective. Since $\operatorname{index} P^{M_p} = \dim \ker P^{M_p} - \dim \operatorname{coker} P^{M_p} = \operatorname{index} P^{L_p}$, it follows that both mappings $\ker P^{M_p} \to \ker P^{L_p}$ and $\operatorname{coker} P^{M_p} \to \operatorname{coker} P^{L_p}$ are isomorphisms. Consequently the inductive limit

$$P(x, d/dx)\colon \mathscr{E}^*(K) \to \mathscr{E}^*(K)$$

has the kernel isomorphic to $\ker P^{M_p}$ and the cokernel isomorphic to $\operatorname{coker} P^{M_p}$.

Next let $L \supset K$ be another compact interval in Ω and consider the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \ker P_K \arrow[r] & \mathscr{E}^*(K) \arrow[r, "P(x,d/dx)"] & \mathscr{E}^*(K) \arrow[r] & \operatorname{coker} P_K \arrow[r] & 0 \\ 0 \arrow[r] & \ker P_L \arrow[r] \arrow[u] & \mathscr{E}^*(L) \arrow[r, "P(x,d/dx)"] \arrow[u] & \mathscr{E}^*(L) \arrow[r] \arrow[u] & \operatorname{coker} P_L \arrow[r] \arrow[u] & 0. \end{tikzcd}$$

The restriction mapping $\mathscr{E}^*(L) \to \mathscr{E}^*(K)$ is surjective by the Whitney type extension theorem [6, Theorem 4.4]. Hence the mapping $\operatorname{coker} P_L \to \operatorname{coker} P_K$ is surjective. Since $L \backslash K$ has no singular points, it follows from Lemma 2 that the mapping $\ker P_L \to \ker P_K$ is injective. Therefore the mappings $\ker P_L \to \ker P_K$ and $\operatorname{coker} P_L \to \operatorname{coker} P_K$ are both isomorphic as above. Employing the Mittag–Leffler argument [3, Lemma 3] relative to the discrete metric, we find that the projective limit

$$P(x, d/dx)\colon \mathscr{E}^*(\Omega) \to \mathscr{E}^*(\Omega)$$

has the kernel isomorphic to $\ker P_K$ and the cokernel isomorphic to $\operatorname{coker} P_K$. In particular, we have

$$\begin{aligned}\operatorname{index}(P\colon \mathscr{E}^*(\Omega) \to \mathscr{E}^*(\Omega)) &= \dim \ker(P\colon E^{M_p}(K) \to E^{M_p}(K)) \\ &\quad -\dim \operatorname{coker}(P\colon E^{M_p}(K) \to E^{M_p}(K)) \\ &= m - d = m - \operatorname{ord}_p a_m(x).\end{aligned}$$

When d is greater than m, $P(x, d/dx)(d/dx)^{d-m}\colon \mathscr{E}^*(\Omega) \to \mathscr{E}^*(\Omega)$ and $(d/dx)^{d-m}\colon \mathscr{E}^*(\Omega) \to \mathscr{E}^*(\Omega)$ are linear operators with indices 0 and $d-m$, respectively. Hence $P(x, d/dx)\colon \mathscr{E}^*(\Omega) \to \mathscr{E}^*(\Omega)$ is a continuous linear mapping with index $m-d$ (see [7, Lemma]). A direct proof may also be obtained by starting with $P(x, d/dx)\colon E^{M_{p+d-m}}(K) \to E^{M_p}(K)$.

In any case $P(x, d/dx)\colon \mathscr{E}^*(\Omega) \to \mathscr{E}^*(\Omega)$ is an open mapping onto the image by De Wilde's closed graph theorem.

The proof is similar for $P(x, d/dx)\colon \mathscr{D}^*(\Omega) \to \mathscr{D}^*(\Omega)$ and $\mathscr{D}^*(\Omega_+) \to \mathscr{D}^*(\Omega_+)$. The inductive limit relative to M_p is discussed in the exactly same way. However, the kernels always vanish by Lemma 2 or Theorem 1. Next let $L \supset K$ be compact intervals and consider the diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{D}_K^* & \xrightarrow{P(x,d/dx)} & \mathscr{D}_K^* & \longrightarrow & \operatorname{coker} P_K & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{D}_L^* & \xrightarrow{P(x,d/dx)} & \mathscr{D}_L^* & \longrightarrow & \operatorname{coker} P_L & \longrightarrow & 0, \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{D}_K^*(\Omega_+) & \xrightarrow{P(x,d/dx)} & \mathscr{D}_K^*(\Omega_+) & \longrightarrow & \operatorname{coker} P_K & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{D}_L^*(\Omega_+) & \xrightarrow{P(x,d/dx)} & \mathscr{D}_L^*(\Omega_+) & \longrightarrow & \operatorname{coker} P_L & \longrightarrow & 0. \end{array}$$

$\operatorname{coker} P_K$ and $\operatorname{coker} P_L$ have the same dimension and the mapping $\operatorname{coker} P_K \to \operatorname{coker} P_L$ is injective because if $f \in \mathscr{D}_K^*$ (resp. $\mathscr{D}_K^*(\Omega_+)$) is equal to $P(x, d/dx)u$ for a $u \in \mathscr{D}_L^*$ (resp. $\mathscr{D}_L^*(\Omega_+)$), then $u \in \mathscr{D}_K^*$ (resp. $\mathscr{D}_K^*(\Omega_+)$) by Therem 1. Hence the mapping $\operatorname{coker} P_K \to \operatorname{coker} P_L$ is an isomorphism in each case. Consequently the inductive limits

$$P(x, d/dx)\colon \mathscr{D}^*(\Omega) \to \mathscr{D}^*(\Omega),$$
$$P(x, d/dx)\colon \mathscr{D}^*(\Omega_+) \to \mathscr{D}^*(\Omega_+)$$

have 0 kernels and cokernels of dimension $m+d$ and d, respectively.

The care for the case $d > m$ and the proof of topological homomorphisms are the same as before. This completes the proof of Theorem 3.

The above proof is based on Lemmas 1–5. All the lemmas hold more or less trivially for the space $\mathscr{E}^*(0)$ of formal power series. Hence we obtain the following.

THEOREM 4. *Suppose that $P(x, d/dx)$ is a differential operator of order m with coefficients $a_i(x) \in \mathscr{E}^*(0)$ which satisfies the irregularity condition. Then $P(x, d/dx)\colon \mathscr{E}^*(0) \to \mathscr{E}^*(0)$ is a topological homomorphism with the index*

$$\operatorname{index}(P\colon \mathscr{E}^*(0) \to \mathscr{E}^*(0)) = m - \operatorname{ord}_0 a_m(x). \tag{3.43}$$

In particular, the differential operator $P(x, d/dx)\colon \mathscr{E}^*(\Omega) \to \mathscr{E}^*(\Omega)$ of Theorem 3 has the same index as $P(x, d/dx)\colon \mathscr{E}^*(0) \to \mathscr{E}^*(0)$. On the other hand, Lemma 4 asserts that $P(x, d/dx)\colon \mathscr{E}^*(\Omega)^0 \to \mathscr{E}^*(\Omega)^0$ is injective. Thus applying the snake theorem to the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{E}^*(\Omega)^0 \arrow[r] \arrow[d, "P(x,d/dx)"] & \mathscr{E}^*(\Omega) \arrow[r] \arrow[d, "P(x,d/dx)"] & \mathscr{E}^*(0) \arrow[r] \arrow[d, "P(x,d/dx)"] & 0 \\ 0 \arrow[r] & \mathscr{E}^*(\Omega)^0 \arrow[r] & \mathscr{E}^*(\Omega) \arrow[r] & \mathscr{E}^*(0) \arrow[r] & 0, \end{tikzcd}$$

we obtain the following.

THEOREM 5. *Suppose that $P(x, d/dx)$ is a differential operator satisfying the same conditions as Theorem 3. Then,*

$$0 \longrightarrow \mathscr{E}^*(\Omega)^0 \xrightarrow{P(x,d/dx)} \mathscr{E}^*(\Omega)^0 \longrightarrow 0 \tag{3.44}$$

is topologically exact and the mapping $\mathscr{E}^*(\Omega) \to \mathscr{E}^*(0)$ of taking the formal Taylor expansion induces the isomorphisms

$$\ker(P\colon \mathscr{E}^*(\Omega) \to \mathscr{E}^*(\Omega)) \cong \ker(P\colon \mathscr{E}^*(0) \to \mathscr{E}^*(0)), \tag{3.45}$$

$$\operatorname{coker}(P\colon \mathscr{E}^*(\Omega) \to \mathscr{E}^*(\Omega)) \cong \operatorname{coker}(P\colon \mathscr{E}^*(0) \to \mathscr{E}^*(0)). \tag{3.46}$$

In other words, the equation

$$P(x, d/dx)\, u(x) = f(x) \tag{3.47}$$

has a unique solution $u \in \mathscr{E}^*(\Omega)^0$ for any $f \in \mathscr{E}^*(\Omega)^0$.

Any formal solution $u \in \mathscr{E}^*(0)$ of

$$P(x, d/dx)\, u(x) = 0$$

is uniquely extended to a real solution $u \in \mathscr{E}^*(\Omega)$. Given an $f \in \mathscr{E}^*(\Omega)$, (3.47) *has a solution* $u \in \mathscr{E}^*(\Omega)$ *if and only if it has a formal solution* $u \in \mathscr{E}^*(0)$.

Similarly we obtain the following theorem from the exact sequences (2.21), (3.41) and (3.42).

THEOREM 6. *Under the same assumptions as Theorem* 3

$$0 \longrightarrow \mathscr{D}^*_{\bar{\Omega}_-}(\Omega) \xrightarrow{P(x,d/dx)} \mathscr{D}^*_{\bar{\Omega}_-}(\Omega) \longrightarrow \mathbf{C}^m \longrightarrow 0 \tag{3.48}$$

is topologically exact and the exact sequence (2.21) *induces the exact sequence*

$$\begin{aligned} 0 &\to \operatorname{coker}(P: \mathscr{D}^*_{\bar{\Omega}_-}(\Omega) \to \mathscr{D}^*_{\bar{\Omega}_-}(\Omega)) \\ &\to \operatorname{coker}(P: \mathscr{D}^*(\Omega) \to \mathscr{D}^*(\Omega)) \\ &\to \operatorname{coker}(P: \mathscr{D}^*(\Omega_+) \to \mathscr{D}^*(\Omega_+)) \to 0. \end{aligned} \tag{3.49}$$

4. DISTRIBUTION AND ULTRADISTRIBUTION SOLUTIONS

Suppose that 0 is a unique singular point of the linear differential operator $P(x, d/dx)$ of order m with coefficients in $\mathscr{E}^*(\Omega)$ on an open interval Ω and that the irregularity condition is satisfied. Then the formal dual $P'(x, d/dx)$ satisfies the same conditions. Taking the duals of the topologically exact sequences (3.42), (3.41), (3.48) and (3.49) with P replaced by P', we obtain the following comutative diagram of exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathbf{C}^{\operatorname{ord}_0 a_m(x)} \arrow[r] \arrow[d] & \mathscr{D}^{*\prime}_{\bar{\Omega}_+}(\Omega) \arrow[r, "P(x,d/dx)"] \arrow[d] & \mathscr{D}^{*\prime}_{\bar{\Omega}_+}(\Omega) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbf{C}^{m+\operatorname{ord}_0 a_m} \arrow[r] \arrow[d] & \mathscr{D}^{*\prime}(\Omega) \arrow[r, "P(x,d/dx)"] \arrow[d] & \mathscr{D}^{*\prime}(\Omega) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbf{C}^m \arrow[r] \arrow[d] & \tilde{\mathscr{D}}^{*\prime}(\Omega_-) \arrow[r, "P(x,d/dx)"] \arrow[d] & \tilde{\mathscr{D}}^{*\prime}(\Omega_-) \arrow[r] \arrow[d] & 0. \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{4.1}$$

The middle row says that the equation

$$P(x, d/dx)\,u(x) = f(x) \tag{4.2}$$

always has a solution $u \in \mathscr{D}^{*\prime}(\Omega)$ for any data $f \in \mathscr{D}^{*\prime}(\Omega)$ and that there are $m + \operatorname{ord}_0 a_m(x)$ linearly independent solutions $u \in \mathscr{D}^{*\prime}(\Omega)$ of the homogeneous equation

$$P(x, d/dx)\,u(x) = 0. \tag{4.3}$$

The first column says that $\operatorname{ord}_0 a_m(x)$ solutions among them can be chosen so as to have support in Ω_+. By the same reason there are $\operatorname{ord}_0 a_m(x)$ linearly independent homogeneous solutions $u \in \mathscr{D}^{*\prime}(\Omega)$ with support in Ω_-.

Comparing the last row of (4.1) with the exact sequence

$$0 \longrightarrow \mathbf{C}^m \longrightarrow \mathscr{D}^{*\prime}(\Omega_-\backslash\{0\}) \xrightarrow{P(x,d/dx)} \mathscr{D}^{*\prime}(\Omega_-\backslash\{0\}) \longrightarrow 0$$

of Theorem 1, we find that every homogeneous solution $u \in \mathscr{D}^{*\prime}(\Omega_-\backslash\{0\})$ on $\Omega_-\backslash\{0\}$ is extendable across 0. Moreover, the first column of (4.1) shows that u can be continued to a homogeneous solution $\tilde{u} \in \mathscr{D}^{*\prime}(\Omega)$ on Ω.

Similarly every solution $u \in \mathscr{D}^{*\prime}(\Omega_+\backslash\{0\})$ of (4.3) on $\Omega_+\backslash\{0\}$ can be continued to a solution $\tilde{u} \in \mathscr{D}^{*\prime}(\Omega)$ on Ω.

Our main theorem is the following (cf. Theorems A, B and C in the introduction).

THEOREM 7. *Suppose that $P(x, d/dx)$ is a linear differential operator of order m with coefficients $a_i(x) \in \mathscr{E}^*(\Omega)$ on an open interval Ω. We assume that the singular points of $P(x, d/dx)$ are isolated and that $P(x, d/dx)$ satisfies the irregularity condition* (2.10) *or* (2.11) *or* (2.12), *at every singular point.*

Then for any $f \in \mathscr{D}^{*\prime}(\Omega)$ Eq. (4.2) *has a solution $u \in \mathscr{D}^{*\prime}(\Omega)$.*

The homogeneous equation (4.3) *has*

$$m + \sum_{x \in \Omega} \operatorname{ord}_x a_m(x) \tag{4.4}$$

linearly independent solutions $u \in \mathscr{D}^{*\prime}(\Omega)$.

If $f \in \mathscr{D}^{*\prime}(\Omega)$, then any solution $u_1 \in \mathscr{D}^{*\prime}(\Omega_1)$ of (4.2) *on an open subinterval Ω_1 of Ω can be continued to a solution $u \in \mathscr{D}^{*\prime}(\Omega)$ on Ω.*

Proof. We order the singular points x_j so that

$$\cdots < x_{-m} < \cdots < x_{-2} < x_{-1} < x_0 < x_1 < x_2 < \cdots < x_n < \cdots$$

and choose a partition of unity $\sum \chi_j(x) = 1$ of class $*$ so that $\operatorname{supp} \chi_j$ is a compact set in the interval (x_{j-1}, x_{j+1}). However, if x_{-m} is the smallest singular point, then we set $\chi_{-m}(x) = 1$ for $x < x_{-m}$ and if x_n is the greatest singular point, then we set $\chi_n(x) = 1$ for $x > x_n$.

Let $v_j \in \mathscr{D}^{*\prime}((x_{j-1}. x_{j+1}))$ be a solution of

$$P(x, d/dx)\, v_j(x) = \chi_j(x)\, f(x) \tag{4.5}$$

on (x_{j-1}, x_{j+1}). There are homogeneous solutions w_j^- on (x_{j-1}, x_j) and w_j^+ on (x_j, x_{j+1}) which coincide with v_j near x_{j-1} and x_{j+1}, respectively. As we remarked earlier they can be continued to homogeneous solutions $\tilde{w}_j^-$ and $\tilde{w}_j^+ \in \mathscr{D}^{*\prime}(\Omega)$ on Ω. Hence v_j is also continued to a solution $\tilde{v}_j$ of (4.5) on Ω. Now define $u_j \in \mathscr{D}^{*\prime}(\Omega)$ by

$$\begin{aligned} u_j &= \tilde{v}_j - \tilde{w}_j^-, \qquad j \geqslant 0, \\ &= \tilde{v}_j - \tilde{w}_j^+, \qquad j < 0. \end{aligned}$$

Then $u = \sum u_j$ is a locally finite sum and gives a solution $u \in \mathscr{D}^*(\Omega)$ of (4.2).

A basis of the space $\ker(P: \mathscr{D}^{*\prime}(\Omega) \to \mathscr{D}^{*\prime}(\Omega))$ of homogeneous solutions is constructed in the following way.

Let $v_1, \ldots, v_m$ be a basis of the homogeneous solutions on (x_{-1}, x_0). They can be continued to homogeneous solutions $\tilde{v}_1, \ldots, \tilde{v}_m$ on Ω.

If $j \geqslant 0$, we choose a basis $w_{j1}, \ldots, w_{jd_j} \in \mathscr{D}^{*\prime}_{[x_j, x_{j+1})}((x_{j-1}, x_{j+1}))$ of the homogeneous solutions on (x_{j-1}, x_{j+1}) with support in $[x_j, x_{j+1})$, where $d_j = \operatorname{ord}_{x_j} a_m(x)$, and extend them to homogeneous solutions $\tilde{w}_{j1}, \ldots, \tilde{w}_{jd_j}$ on Ω so that they have support in $\{x \in \Omega; x \geqslant x_j\}$.

Similarly we construct for $j < 0$ homogeneous solutions $\tilde{w}_{j1}, \ldots, \tilde{w}_{jd_j}$ on Ω with support in $\{x \in \Omega; x \leqslant x_j\}$ such that their restrictions to (x_{j-1}, x_{j+1}) form a basis of the homogeneous solutions with support in $(x_{j-1}, x_j]$.

Then $\tilde{v}_1, \ldots, \tilde{v}_m$, $\tilde{w}_{01}, \ldots, \tilde{w}_{0d_0}$, $\tilde{w}_{11}, \ldots, \tilde{w}_{1d_1}, \ldots, \tilde{w}_{-11}, \ldots, \tilde{w}_{-1d_{-1}}, \ldots$ are clearly linearly independent.

Let $u \in \mathscr{D}^{*\prime}(\Omega)$ be an arbitrary homogeneous solution. Its restriction to (x_{-1}, x_0) is a linear combination of $v_1, \ldots, v_m$, so that $u - (c_1 \tilde{v}_1 + \cdots + c_m \tilde{v}_m)$ is a homogeneous solution on Ω with support in $\{x \in \Omega; x \leqslant x_{-1}$ or $x \geqslant x_0\}$. Similarly subtracting a linear combination $c_{01} \tilde{w}_{01} + \cdots + c_{0d_0} \tilde{w}_{0d_0}$, we can make it a homogeneous solution with support in $\{x \in \Omega; x \leqslant x_{-1}$ or $x \geqslant x_1\}$. Then we subtract a linear combination $c_{-11} \tilde{w}_{-11} + \cdots + c_{-1d_{-1}} \tilde{w}_{-1d_{-1}}$ to make it a homogeneous solution with support in $\{x \in \Omega; x \leqslant x_{-2}$ or $x \geqslant x_1\}$ and so on. Thus u can be expressed as a locally finite sum

$$\sum_{i=1}^{m} c_i \tilde{v}_i + \sum_j \sum_{k=1}^{d_j} c_{jk} \tilde{v}_{jk}.$$

Let Ω_1 be an open subinterval of Ω. We may assume without loss of generality that $\Omega_1 \cap (x_{-1}, x_0) \neq \varnothing$. Then the same proof shows that every homogeneous solution $u_1 \in \mathscr{D}^{*\prime}(\Omega_1)$ is a locally finite sum

$$\sum_{i=1}^{m} c_i \tilde{v}_i + \sum_{x_j \in \Omega_1} \sum_{k=1}^{d_j} c_{jk} \tilde{v}_{jk}$$

on Ω_1, so that the sum gives an extension to a homogeneous solution on Ω.

Every $f \in \mathscr{D}^{*\prime}(\Omega)$ is written $P(x, d/dx)\, u_0$ for a $u_0 \in \mathscr{D}^{*\prime}(\Omega)$. If $u_1 \in \mathscr{D}^{*\prime}(\Omega_1)$ is a solution of $Pu_1 = f$ on Ω_1, then $u_1 - u_0$ is a homogeneous solution on Ω_1, which can be continued to a homogeneous solution u_2 on Ω, and hence u_1 is continued to the solution $u = u_0 + u_2$ on Ω.

This completes the proof of Theorem 7.

To formulate an analogue of Theorems E, F and G, we define the following order relation between classes $\varnothing$, (s) and $\{s\}$:

$$\phi > \{s\} > (s) > \{t\} \qquad \text{for} \quad s > t > 1. \tag{4.6}$$

Thus we have $* \geqslant \dagger$ if and only if $\mathscr{E}^*(\Omega) \supset \mathscr{E}^\dagger(\Omega)$, and if and only if $\mathscr{D}^{*\prime}(\Omega) \subset \mathscr{D}^{\dagger\prime}(\Omega)$ for *a* (and any) non empty open set Ω.

THEOREM 8. *Suppose that $P(x, d/dx)$ is a linear differential operator of order m with coefficients $a_i(x) \in \mathscr{E}^\dagger(\Omega)$ on an open interval Ω and that $* \geqslant \dagger$.*

If the singular points of $P(x, d/dx)$ are isolated and if $P(x, d/dx)$ satisfies the irregularity condition (2.10) *or* (2.11) *or* (2.12) *with respect to $*$ at every singular point, then any solution $u \in \mathscr{D}^{\dagger\prime}(\Omega)$ of* (4.2) *belongs to $\mathscr{D}^{*\prime}(\Omega)$ whenever the data $f \in \mathscr{D}^{*\prime}(\Omega)$.*

Proof. A posteriori the operator $P(x, d/dx)$ satisfies the irregularity condition with respect to $\dagger$ at every singular point. Hence the homogeneous equation (4.3) has the same number of linearly independent solutions $u \in \mathscr{D}^{\dagger\prime}(\Omega_1)$ as the number of linearly independent solutions $u \in \mathscr{D}^{*\prime}(\Omega_1)$ on every relatively compact open subinterval. Consequently every homogeneous solution $u \in \mathscr{D}^{\dagger\prime}(\Omega)$ belongs to $\mathscr{D}^{*\prime}(\Omega)$.

On the other hand, there is a solution $u_0 \in \mathscr{D}^{*\prime}(\Omega)$ of $Pu_0 = f$. Hence $u = (u - u_0) + u_0$ belongs to $\mathscr{D}^{*\prime}(\Omega)$.

This completes the proof of Theorem 8.

Besides (2.20) we have the topologically exact sequence

$$0 \longrightarrow \mathscr{D}^*(\Omega)^0 \longrightarrow \mathscr{D}^*(\Omega) \longrightarrow \mathscr{E}^*(0) \longrightarrow 0 \tag{4.7}$$

[6, Theorem 4.4]. Here

$$\mathscr{D}^*(\Omega)^0 = \mathscr{D}^*_{\Omega\_}(\Omega) \oplus \mathscr{D}^*_{\Omega\_}(\Omega),$$

so that we have the topologically exact sequence

$$0 \to \mathscr{D}^{*\prime}_{\{0\}} \to \mathscr{D}^{*\prime}(\Omega) \to \tilde{\mathscr{D}}^{*\prime}(\Omega_-) \oplus \tilde{\mathscr{D}}^{*\prime}(\Omega_+) \to 0 \tag{4.8}$$

as the strong dual of (4.7), where $\mathscr{D}^{*\prime}_{\{0\}}$ is the space of all ultradistributions of class $*$ on $\mathbf{R}$ with support in $\{0\}$.

Thus the following theorem follows from Theorems 3 and 4.

THEOREM 9. *Suppose that $P(x, d/dx)$ is a linear differential operator of order m with coefficients $a_i(x) \in \mathscr{E}^*(\Omega)$ on an open interval Ω and that $0 \in \Omega$ is a unique singular point at which the irregularity condition* (2.10) *or* (2.11) *or* (2.12) *is satisfied. Then $P(x, d/dx)\colon \mathscr{D}^{*\prime}_{\{0\}} \to \mathscr{D}^{*\prime}_{\{0\}}$ is a topological homomorphism with the index*

$$\operatorname{index}(P\colon \mathscr{D}^{*\prime}_{\{0\}} \to \mathscr{D}^{*\prime}_{\{0\}}) = \operatorname{ord}_0 a_m(x) - m \tag{4.9}$$

and

$$\begin{aligned} 0 &\to \ker(P\colon \mathscr{D}^{*\prime}_{\{0\}} \to \mathscr{D}^{*\prime}_{\{0\}}) \to \ker(P\colon \mathscr{D}^{*\prime}(\Omega) \to \mathscr{D}^{*\prime}(\Omega)) \\ &\to \ker(P\colon \tilde{\mathscr{D}}^{*\prime}(\Omega_-) \to \tilde{\mathscr{D}}^{*\prime}(\Omega_-)) \oplus \ker(P\colon \tilde{\mathscr{D}}^{*}(\Omega_+) \to \tilde{\mathscr{D}}(\Omega_+)) \\ &\to \operatorname{coker}(P\colon \mathscr{D}^{*\prime}_{\{0\}} \to \mathscr{D}^{*\prime}_{\{0\}}) \to 0 \end{aligned} \tag{4.10}$$

is exact.

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